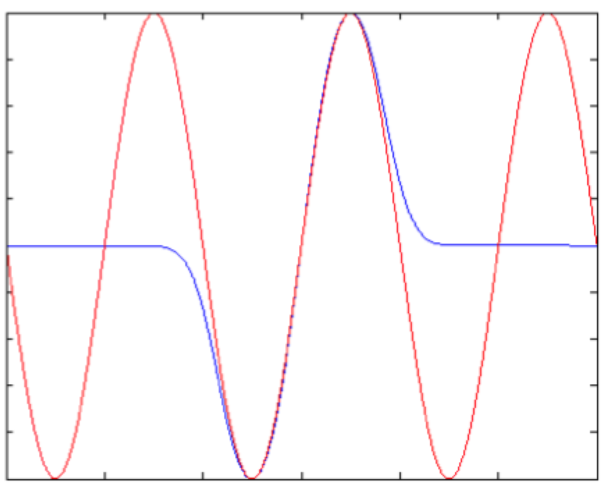
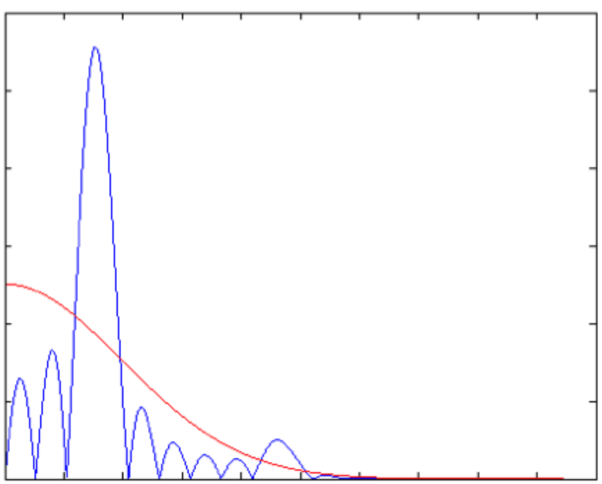
Spectral Methods for Neural Computation

Michael Lindsey
Boahen Lab Meeting
January 28, 2014





1. Outline

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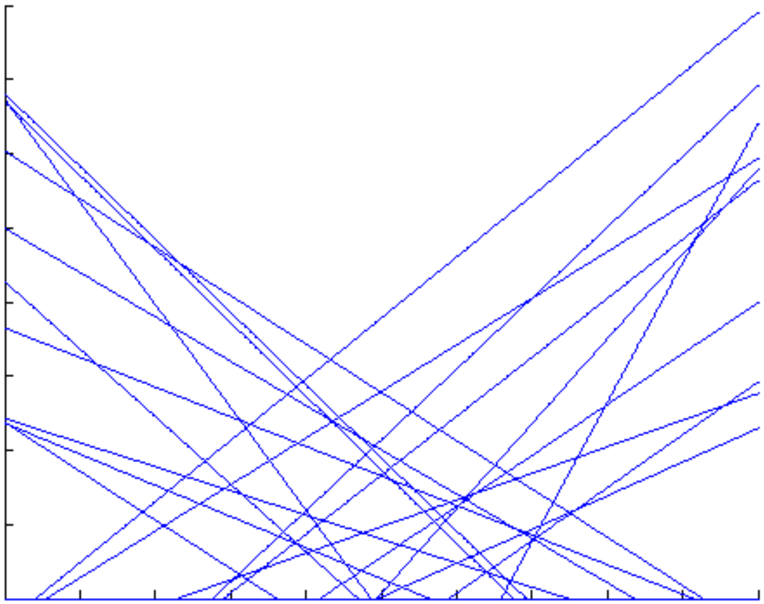
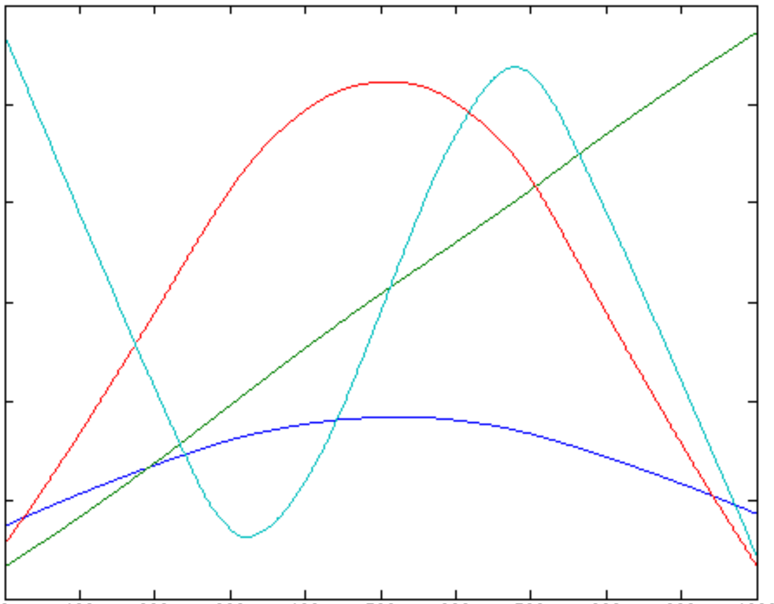
- (Analogous method for computing polynomials)

- (Application: numerical integration)

2. A Motivating Empirical Result



'Hinge' tuning curves



'Polynomial' basis

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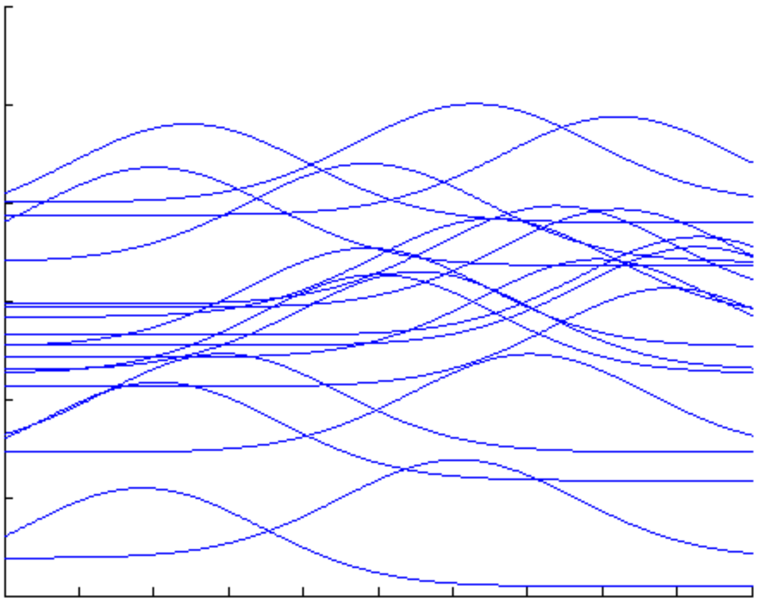
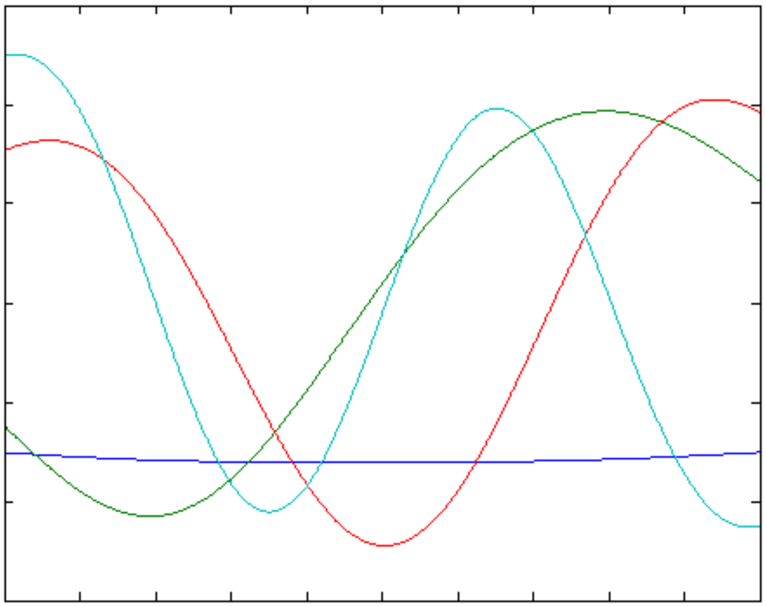


'Gaussian' tuning curves



'Fourier' basis

3. A Shot in the Dark

- Try adding up translated ($\pm$) Gaussian functions with extrema aligned with local extrema of sinusoid

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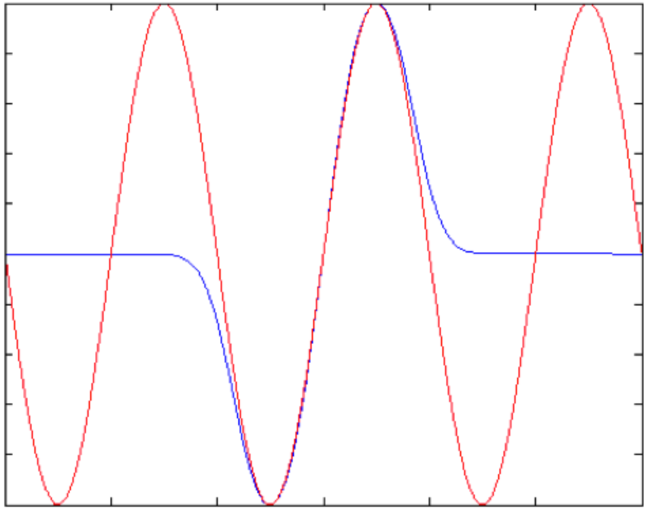
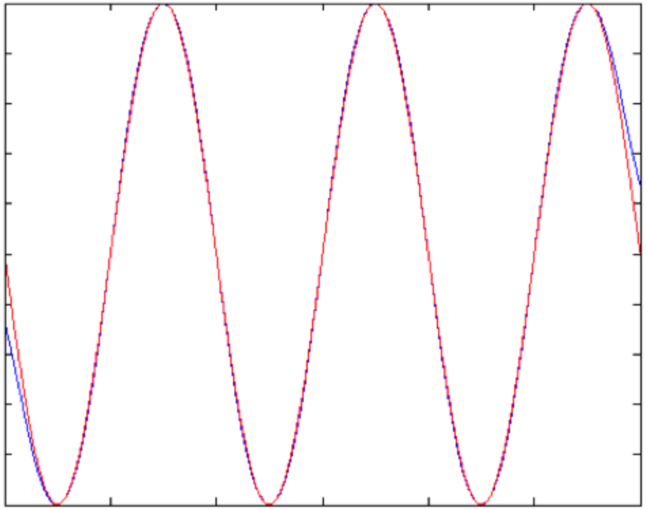
- Try adding up translated (±) Gaussian functions with extrema aligned with local extrema of sinusoid

- Surprising result! But it's no accident...





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- Property 3: $\mathcal{F}$ and $\mathcal{F}^{-1}$ map Schwartz functions to Schwartz functions (in fact, FT of Gaussian is Gaussian)

- Notionally: smoothness in spatial domain $\leftrightarrow$ decay in frequency domain (also, decay in spatial domain $\leftrightarrow$ smoothness in frequency domain)

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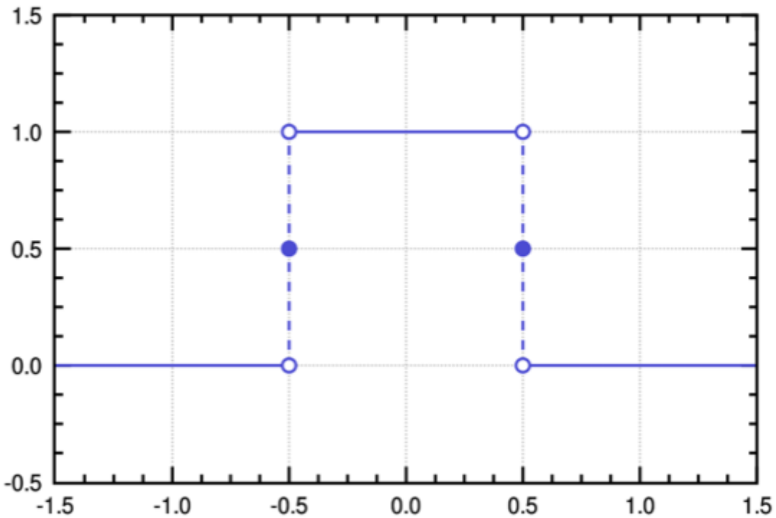
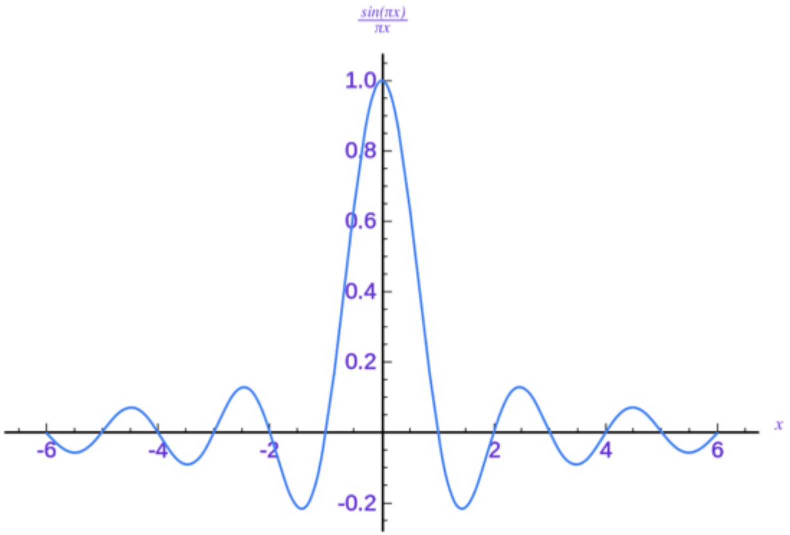


Discontinuity



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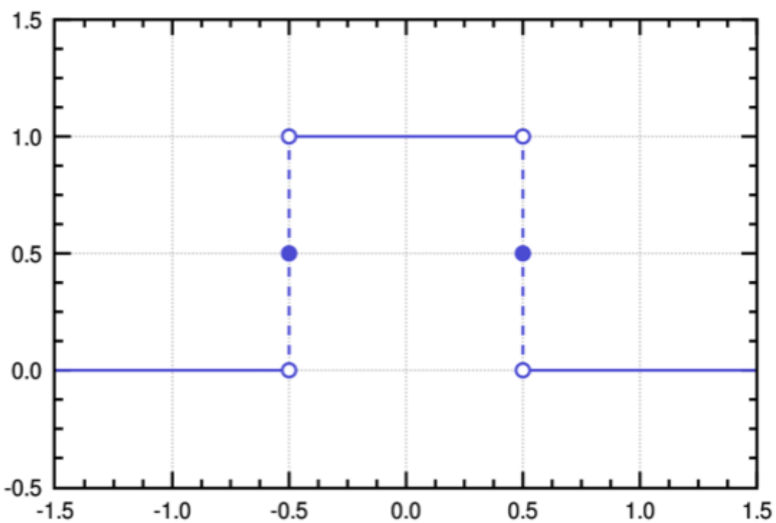

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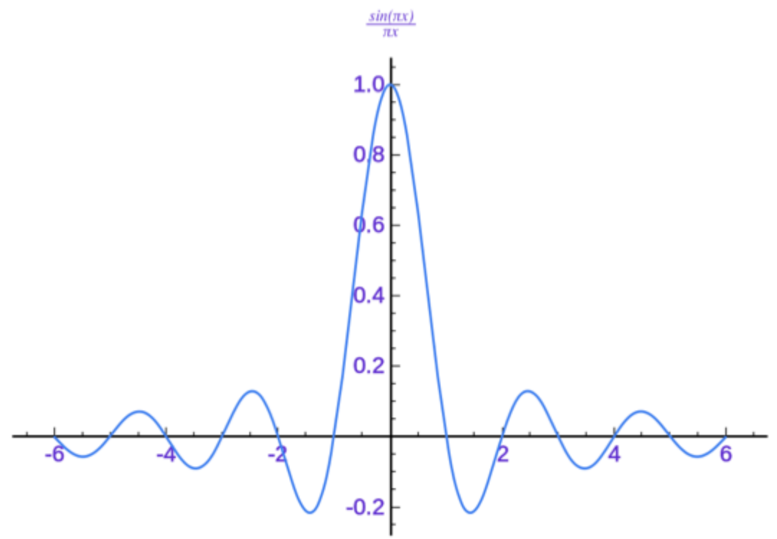

Slow decay

-Property 4 (scaling): If $f_a(x) = f(\frac{x}{a})$, then $\widehat{f_a}(\omega) = |a|\widehat{f}(a\omega)$

5. Precise Statement for Constructing Sinusoids

- Let g be a Schwartz function. Let $x_k^{(+)} = 1 + 4k$, $x_k^{(-)} = -1 + 4k$. Let $g_k^{(+)}(x) = g(x - x_k^{(+)})$ and $g_k^{(-)}(x) = g(x - x_k^{(-)})$

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- Let $f_N = \left(g_0^{(+)} - g_0^{(-)}\right) + \sum_{k=1}^{N} \left(g_k^{(+)} - g_k^{(-)} + g_{-k}^{(+)} - g_{-k}^{(-)}\right)$

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- Then for all $x \in \mathbb{R}$, as $N \to \infty$,

$$f_N(x) \to \sum_{k=0}^{\infty} (-1)^k \left[a_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)x\right) - b_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)x\right)\right]$$

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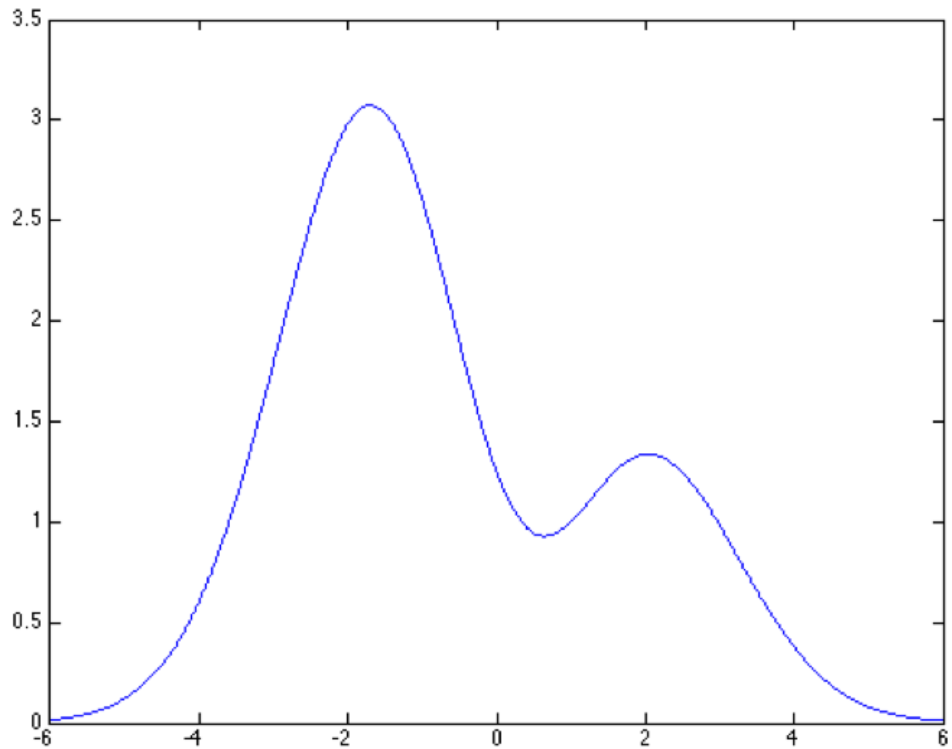
$$

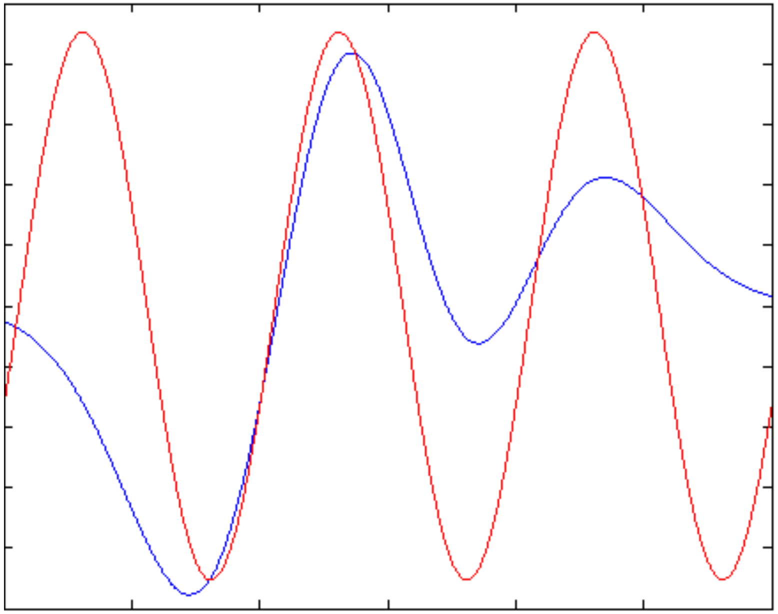
where $a_k = \Re\left(\widehat{g}\left(\frac{\pi}{2} + k\pi\right)\right)$, $b_k = \Im\left(\widehat{g}\left(\frac{\pi}{2} + k\pi\right)\right)$ for all k.

6. A Surprising Consequence

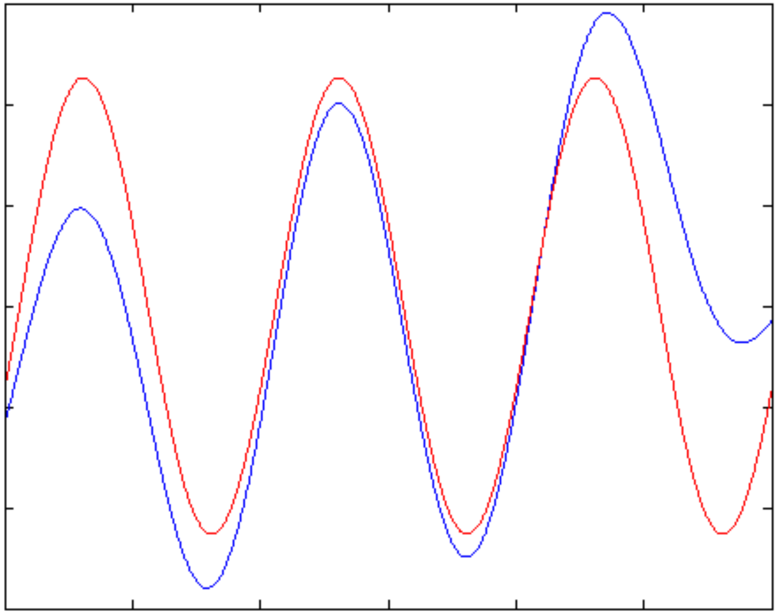
- We do not require that the tuning curve g have a single local extremum



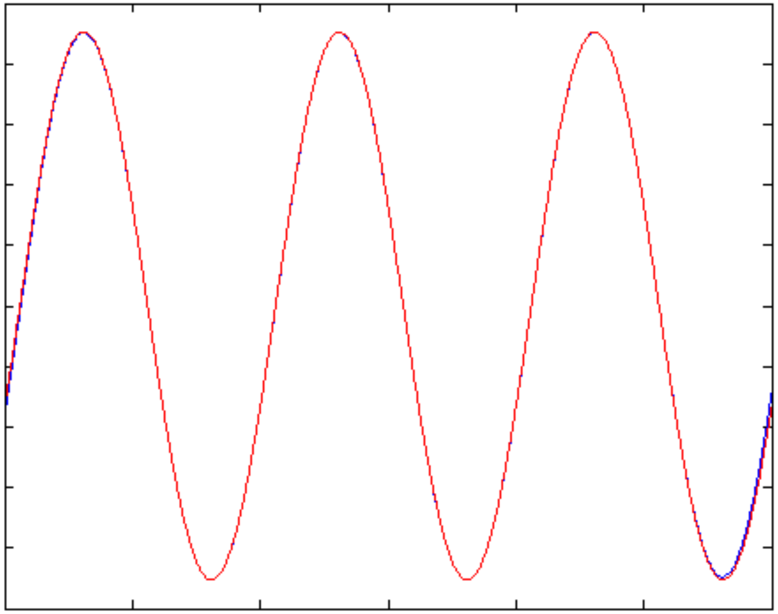
Weird tuning curve



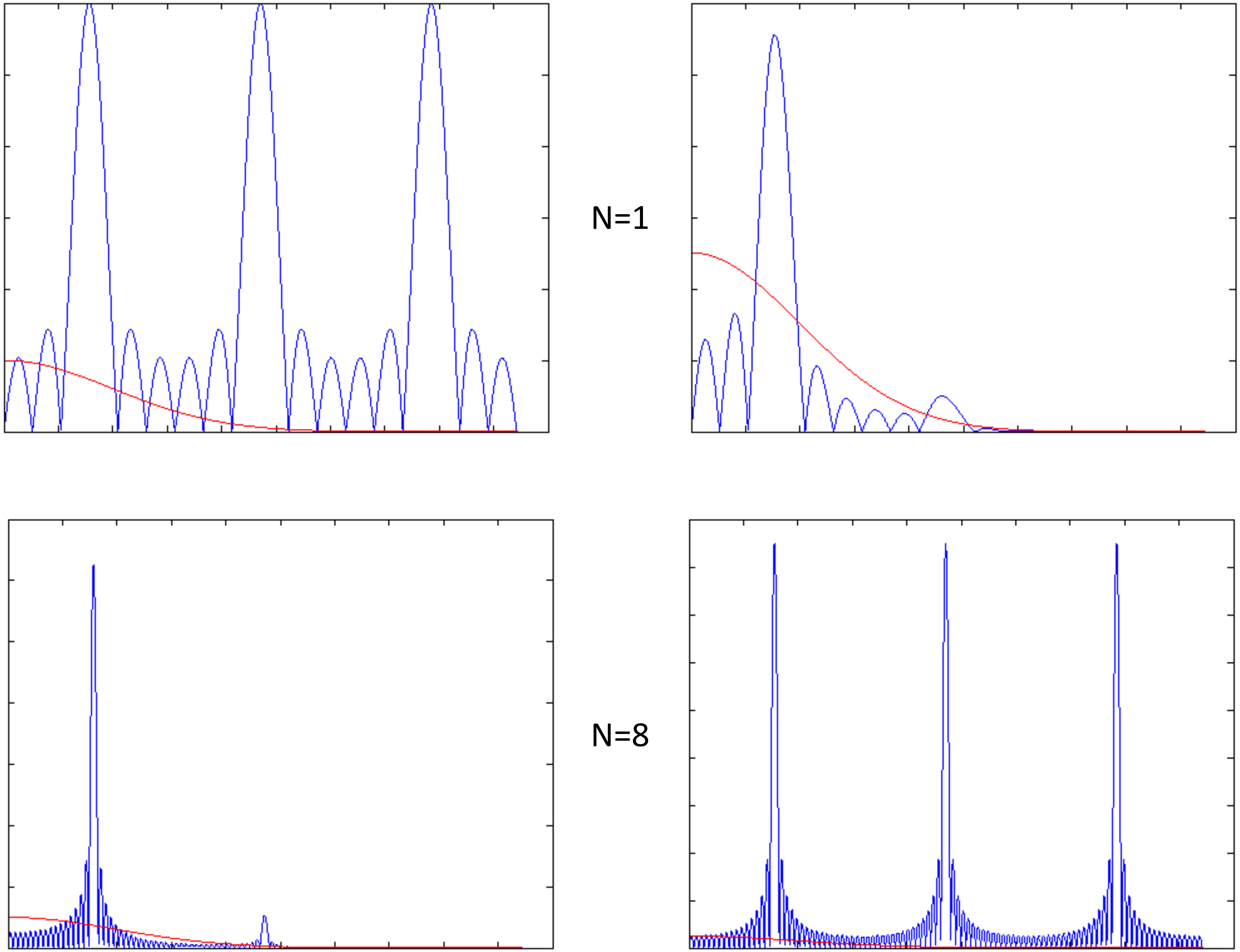
2 neurons



6 neurons



10 neurons


N=1
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- However, cannot guarantee that $a_0 \gg a_k$ for all $k \geq 1$. How to guarantee rapidly decaying Fourier transform?

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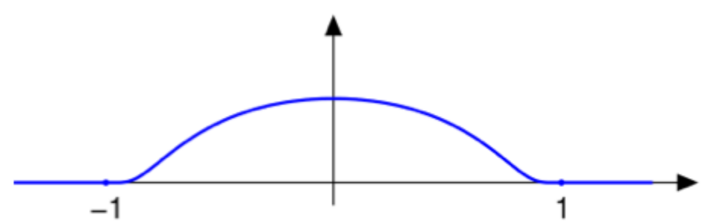

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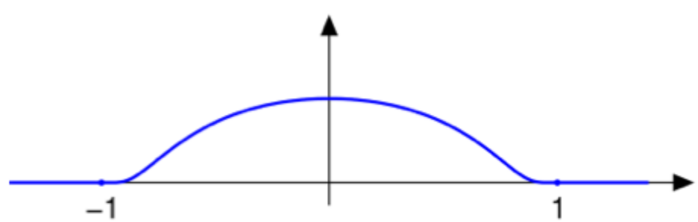
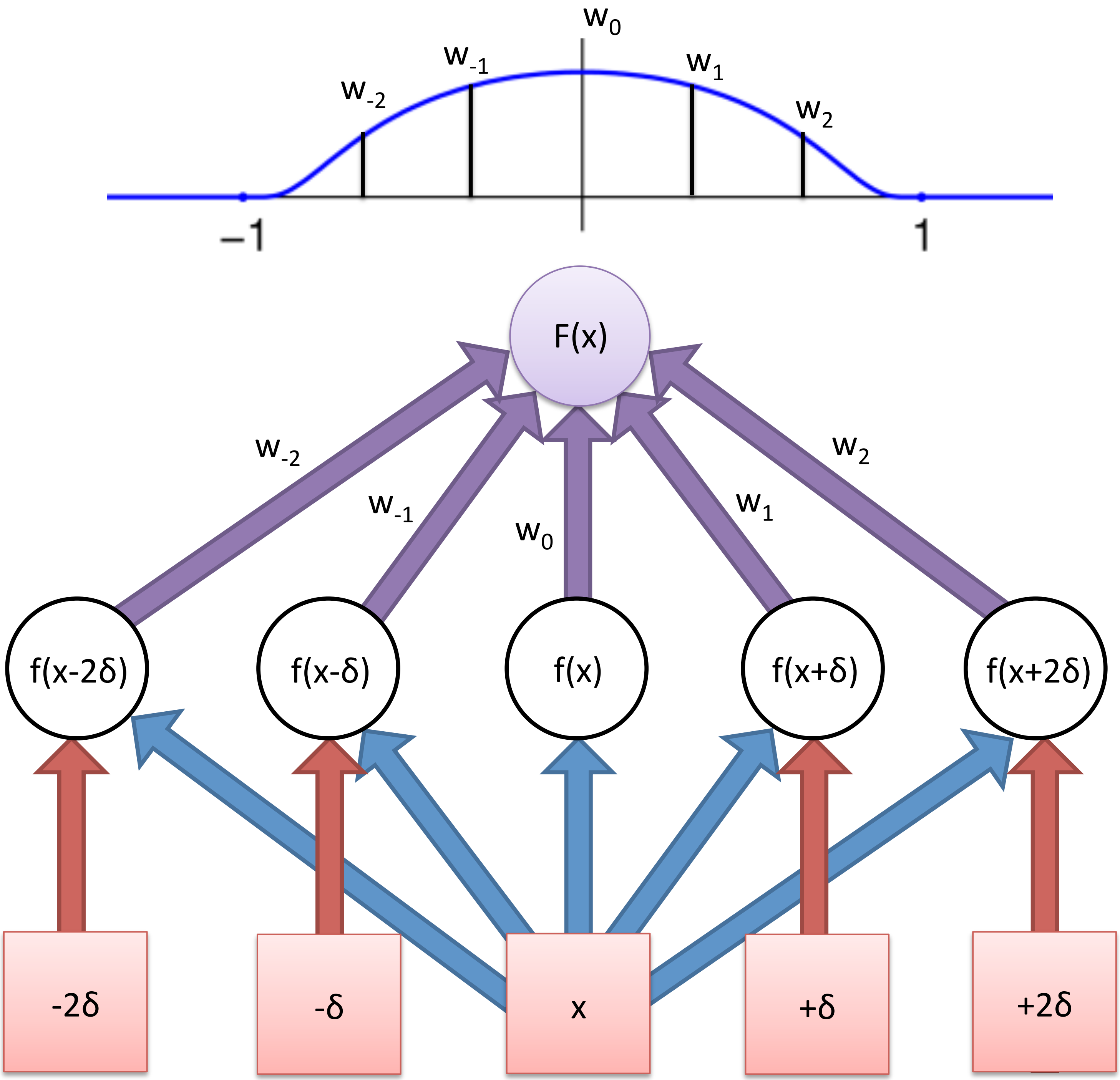


- A discrete mollification can be carried out by a simple neural network:

$$\widetilde{f}(x) = \left(\sum_{j=-n+1}^{n-1} \varphi\left(\frac{j}{n}\right)\right)^{-1} \sum_{j=-n+1}^{n-1} \varphi\left(\frac{j}{n}\right) f\left(x - j\delta\right)$$



w_0
w_{-1}
w_1
w_{-2}
w_2
−1
1
F(x)
w_{-2}
w_2
w_{-1}
w_1
w_0
f(x-2δ)
f(x-δ)
f(x)
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-δ
x
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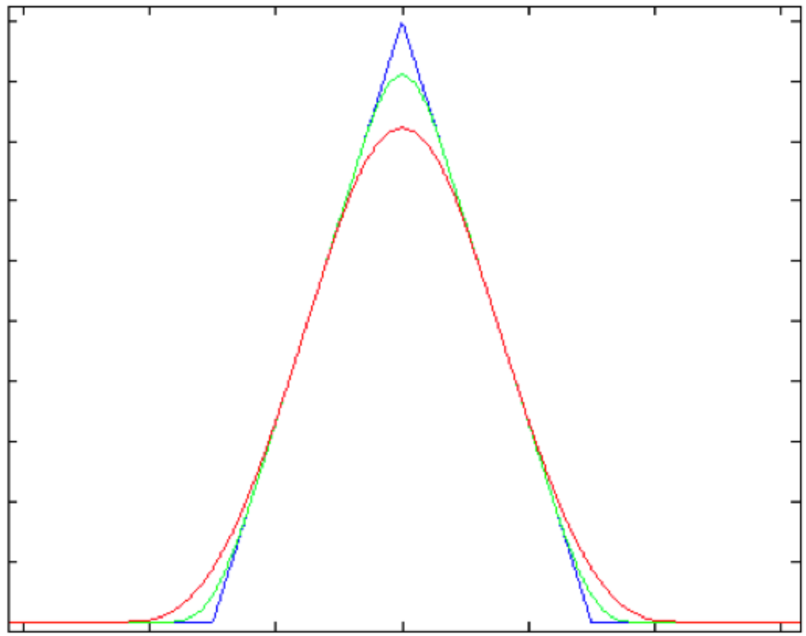
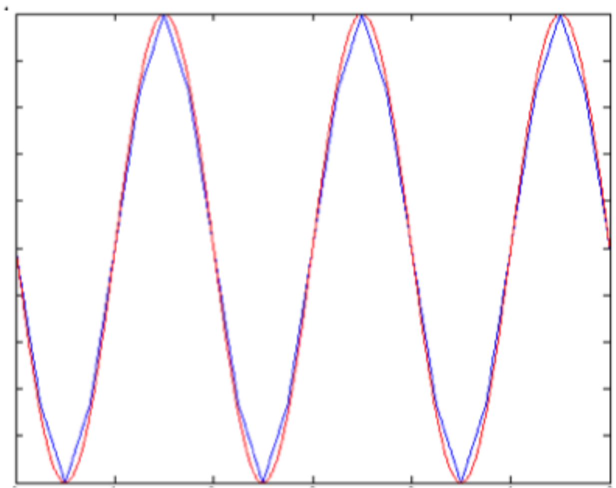


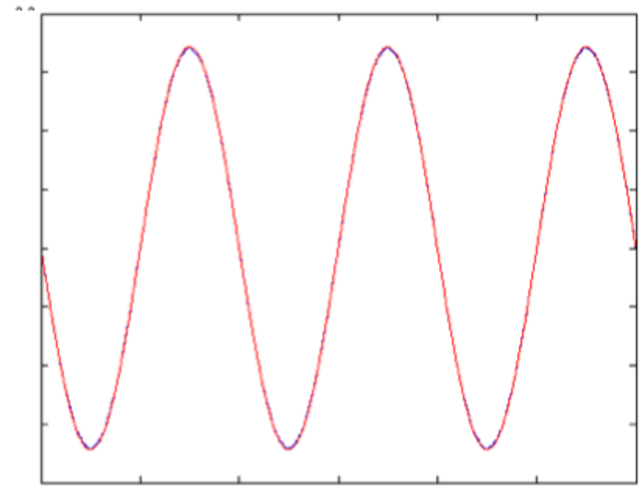
Mollified hat functions obtained from above procedure (with $\delta = 0.1$)

Blue: no mollification. Green: $n = 4$ (convex combination of 7 hat functions). Red: $n = 8$ (15 hat functions)

Approximation using no mollification (left), mollification with $\delta = 0.3$, $n = 4$ (right)



Without smoothing



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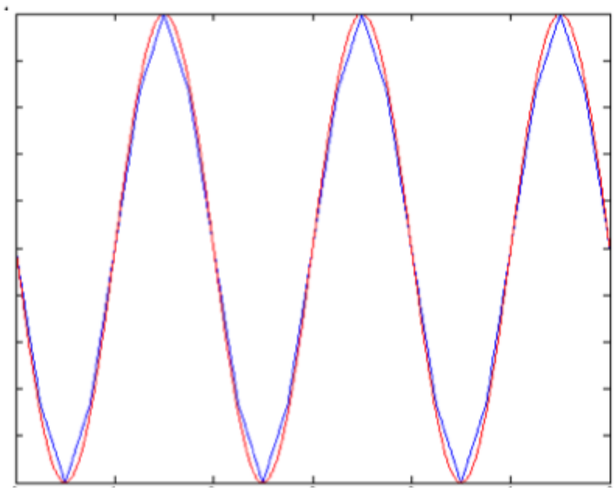

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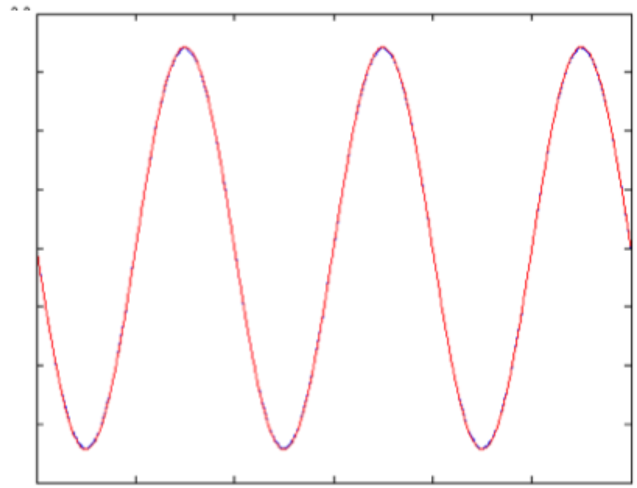

With smoothing

n	0	4	8
L^2 error	1.3×10^{-3}	6.7×10^{-5}	2.9×10^{-5}
L^∞ error	0.0912	0.0065	0.0024

So to approximate one period of a sinusoid, we require about 14 hat-shaped tuning curves (as opposed to 2 Gaussian tuning curves)

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- We may need to choose sample spacing δ smaller for more irregular tuning curve shapes

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$$\sigma^2(f) := \inf_{t_0} \int_{\mathbb{R}} (t - t_0)^2 P(t) dt,$$

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Weyl-Heisenberg Uncertainty Principle:

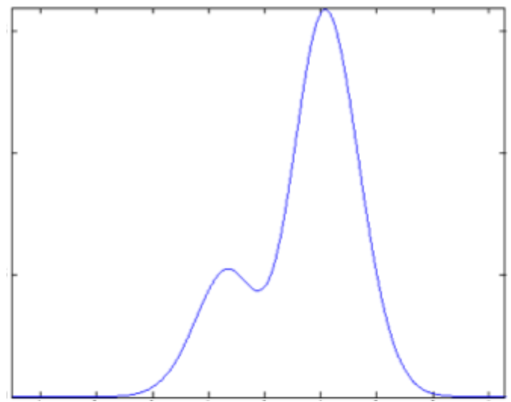
$\sigma(f)\sigma(\widehat{f}) \geq \frac{1}{2}$, with equality if and only if f is a Gaussian

Review

- We can build sinusoids from smooth, rapidly decaying tuning curves
- It's okay if the tuning curves have many peaks
- …but Gaussians are the best
- We can deal with non-smooth tuning curves
- Network structure itself encodes computation
- Robust to modification of tuning curve
- Sinusoids as basis

Application: Statistical Inference

- Take $g(x) = (2x^2 + 0.5)e^{-(x-0.32)^2}$



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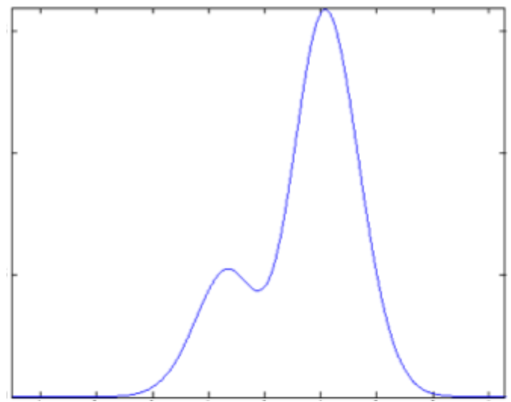
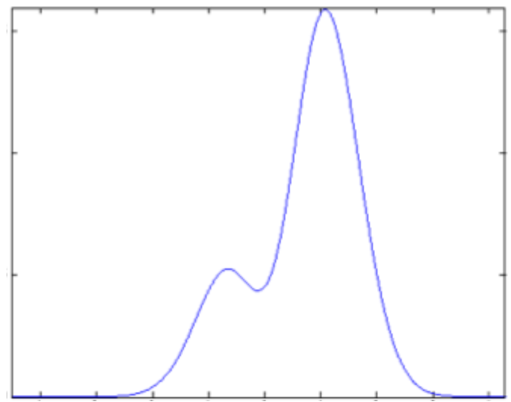


- We approximate the p-th moment of g by $\sum_{n=-3}^{3} n^p g(n)$

$$\sum_{n=-3}^{3} n^p g(n) \quad \int_{\infty}^{\infty} u^p g(u) du$$

Application: Statistical Inference

- Take $g(x) = (2x^2 + 0.5)e^{-(x-0.32)^2}$



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	$\sum_{n=-3}^{3} n^p g(n)$	$\int_{\infty}^{\infty} u^p g(u) du$
$p = 0$	3.02	3.02
$p = 1$	2.11	2.10
$p = 2$	4.32	4.32
$p = 3$	5.24	5.30

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In particular, by modifying g with an appropriate horizontal scaling if necessary, we obtain the approximation (for large enough N) $f_N(x) \approx \sum_{n=0}^{p} c_n x^n$, where $c_n = \int u^{p-n} g(u) du$, so c_n are constants and f_N is approximately a polynomial.

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- Thus we are equipped to do robot control using the above methods with explicit error bounds

Conclusions

- smoothness allows for discrete approach to continuous problems
- spectral intuition
- efficient, robust, general

Future work

- spike-based model
- heterogeneity
- time domain
- hardware-specific considerations