Math 54: Worksheet Solution

April 18

(1) Find the general solution of

$$y''(t) + y(t) = t\cos(t).$$

Solution: The homogeneous equation has general solution $y(t) = C_1\sin(t) + C_2\cos(t)$. Our recipe for guessing a particular solution (method of undetermined coefficients) tells us to guess

$$y_p(t) = t\,[a_0 + a_1 t]\cos(t) + t\,[b_0 + b_1 t]\sin(t).$$

The rest of the details are omitted.

(2) True or false: the set of solutions of the ODE in the last problem is a vector space.

Solution: False, because the ODE is inhomogeneous (so the sum of two solutions is no longer a solution).

(3) Consider the differential equation:

$$y''(t) + y(t) = 0.$$

Derive an equivalent linear system of differential equations in normal form, i.e., in the form $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$.

Solution: Let $x_1(t) = y(t)$, $x_2(t) = x_1'(t) = y'(t)$. Then we have:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= y''(t) = -y(t) = -x_1(t), \end{aligned}$$

i.e.,

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}(t),$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

(4) Consider the system of differential equations:

$$\begin{cases} y_1''(t) + t y_1'(t) - y_2(t) = e^t \\ y_2'(t) + \cos(t) y_1(t) = 0 \end{cases}$$

Derive an equivalent linear system of differential equations in normal form.

Solution: Let $x_1(t) = y_1(t)$, $x_2(t) = x_1'(t) = y_1'(t)$, $x_3(t) = y_2(t)$. Then we have:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= y_1''(t) = -t y_1'(t) + y_2(t) + e^t = -t x_2(t) + x_3(t) + e^t \\ x_3'(t) &= y_2'(t) = -\cos(t) y_1(t) = -\cos(t) x_1(t), \end{aligned}$$

i.e.,

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ -\cos(t) & 0 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$.

(5) Recall that for $\mathbb{R}^n$-valued functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the Wronskian of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is the $\mathbb{R}$-valued function $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det\left(\mathbf{x}_1(t) \ \cdots \ \mathbf{x}_n(t)\right)$.

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are solutions of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on some open interval I. Then which of the following are possible: (1) the Wronskian is zero on all of I, (2) the Wronskian is never zero on I, (3) the Wronskian takes both zero and nonzero values on I.

In each case, what can we conclude about the linear independence or linear dependence of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$?

Solution: Only cases (1) and (2) are possible. In the first case, we have linear dependence. In the second case, we have linear independence.

(6) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2]$ determined by

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}.$$

$\mathbf{x}_1$ and $\mathbf{x}_2$ happen to be solutions of a system of ODEs $\mathbf{x}'(t) = A\mathbf{x}(t)$. Can you figure out what A must be?

Now consider

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

Is it possible that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of some system $\mathbf{x}'(t) = A\mathbf{x}(t)$? How about $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$? How could this be determined from the Wronskian?

Solution: For the first part, the Wronskian is given by $W[\mathbf{x}_1, \mathbf{x}_2](t) = \sin^2(t) + \cos^2(t) = 1$. To figure out A, notice that

$$\mathbf{x}_1'(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}_1(t),$$

so take $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. You can check that $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve $\mathbf{x}'(t) = A\mathbf{x}(t)$.

For the second part, compute $W[\mathbf{x}_1, \mathbf{x}_2](t) = \sin^2(t) - \cos^2(t)$. This function is zero at some points but not all points. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot be solutions of some homogeneous linear system of differential equations (see last problem).

(7) Suppose that $\mathbf{x} : I \to \mathbb{R}^n$ solves $\mathbf{x}'(t) = A\mathbf{x}(t)$, and moreover suppose that we can diagonalize $A = PDP^{-1}$. Define a new function $\mathbf{z} : I \to \mathbb{R}^n$ by $\mathbf{z}(t) = P^{-1}\mathbf{x}(t)$. Verify that $\mathbf{z}$ solves $\mathbf{z}'(t) = D\mathbf{z}(t)$. What is so great about this observation?

Solution: Compute

$$\mathbf{z}'(t) = \left[P^{-1}\mathbf{x}\right]'(t) = P^{-1}\mathbf{x}'(t) = P^{-1}A\mathbf{x}(t) = P^{-1}PDP^{-1}\mathbf{x}(t) = DP^{-1}\mathbf{x}(t) = D\mathbf{z}(t),$$

so we have confirmed that $\mathbf{z}'(t) = D\mathbf{z}(t)$. Since D is diagonal, let $\lambda_1, \ldots, \lambda_n$ be its diagonal entries. Then $\mathbf{z}'(t) = D\mathbf{z}(t)$ simply means that $z_1'(t) = \lambda_1 z_1(t), \ldots, z_n'(t) = \lambda_n z_n(t)$. Each of these equations can be solved easily on its own (they are not coupled to each other). Then once we know $\mathbf{z}(t)$, we can determine $\mathbf{x}(t)$ via the relation $\mathbf{z}(t) = P\mathbf{x}(t)$.